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Seminvariants and Symmetric Functions.

By Capt. P. A. MacMahon, R. A.

In this paper I consider the binary quantic with derived coefficients

$$a_0x^n + na_1x^{n-1}y + n(n-1)a_2x^{n-2}y^2 + n(n-1)(n-2)a_3x^{n-3}y^3 + \ldots$$

or putting y equal to unity, it may be written symbolically thus, viz.

$$(a_0, a_1, a_2 \ldots \between \frac{x^n}{1-\partial_x})$$

where ∂_x stands for $\frac{d}{dx}$.

Where it is convenient a_0 will be supposed unity, and sometimes the literal coefficients will be taken to be $a, b, c, d, \ldots$ An advantage of discussing the equation in this form is that the seminvariant operator

$$\partial_{a_1} + 2a_1\partial_{a_2} + 3a_2\partial_{a_3} + \ldots$$

belonging to the quantic with binomial coefficients $(1, a_1, a_2, \ldots \between x, 1)^n$ becomes in the present case $\partial_{a_1} + a_1\partial_{a_2} + a_2\partial_{a_3} + \ldots$ an operator which, when considered in connection with the equation

$$(1, a_1, a_2, \ldots \between x, 1)^n = 0$$

is such that it reduces all the non-unitary-partition symmetric functions to zero, and further reduces a symmetric function whose partition contains r units to a similar one containing $r-1$ units; consequently the solutions of the partial differential equation $\partial_{a_1} + a_1\partial_{a_2} + a_2\partial_{a_3} + \ldots = 0$ are contained in the tables of symmetric functions which have been already calculated as far as the twelfth degree and amply verified.

We thus arrive at the result that every seminvariant of the equation with derived coefficients is a non-unitary-partition symmetric function of the roots of the equation $(1, a_1, a_2, \ldots \between x, 1)^n = 0$. This paper is based upon the analogy shown to exist between seminvariants and symmetric functions, and it will appear that it gives us a universal method for forming an asyzygetic series of seminvariants of a given weight in a form which discovers the syzygants where such exist.

In lieu of considering the correspondence between the two quantics

$$(a_0, a_1, \ldots \between \frac{x^n}{1-\partial_x}), \quad (a_0, a_1 \ldots \between x, 1)^n$$

we might have discussed in conjunction the quantics with binomial and exponential coefficients, viz. $(a_0, \ldots \between x, 1)^n$ and $(a_0, a_1 \frac{1}{2!} a_2, \frac{1}{3!} a_3, \ldots \between x, 1)^n$, since the non-unitary-partition symmetric functions of the equation formed by equating the latter quantic to zero are seminvariants of the former; but since the converse is not true unless the seminvariants be affected with numerical factors, the method here adopted seemed the better one. The notation is settled so as to be as nearly as possible in accordance with that of Mr. Hammond (*Amer. Jour.*, V, No. 3); viz. $q_2, c_3, q_4, c_5, q_6, \ldots$ are taken to represent the protomorphs or primary groundforms, the suffix denoting at once the extent (*i. e.* rank or weight of highest letter) and the weight of the source.

In Mr. Hammond's paper these are respectively $H, C_3, Q_4, C_5, Q_6, \ldots$ and refer to the quantic with binomial coefficients; any protomorph expression of a seminvariant of the quantic with derived coefficients which involves c's and q's may be transformed so as to belong to the quantic $(a_0, a_1, \ldots \between x, 1)^n$ by means of the equations

$$C_{2m+1} = (2m)!\, c_{2m+1}$$
$$Q_{2m} = m\,(2m-1)!\, q_{2m}$$

the simply numerical factor being rejected; so that the present results may be readily identified with those of Mr. Hammond. We find

$$\begin{aligned}
q_2 &= 2a_2 - a_1^2, \\
c_3 &= 3a_3 - 3a_1 a_2 + a_1^3, \\
q_4 &= 2a_4 - 2a_1 a_3 + a_2^2, \\
c_5 &= 5a_5 - 5a_1 a_4 + a_2 a_3 + 2a_1^2 a_3 - a_1 a_2^2, \\
q_6 &= 2a_6 - 2a_1 a_5 + 2a_2 a_4 - a_3^2, \\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
q_{2m} &= 2a_{2m} - 2a_1 a_{2m-1} + 2a_2 a_{2m-2} - \ldots + (-)^{\frac{1}{2}n} a_m^2, \\
c_{2m+1} &= (2m+1) a_{2m+1} - (2m+1) a_1 a_{2m} + (2m-3) a_2 a_{2m-1} \\
&\quad - (2m-5) a_3 a_{2m-2} + \ldots + a_1 \{ 2a_1 a_{2m-1} - 2a_2 a_{2m-2} \\
&\quad + 2a_3 a_{2m-3} - \ldots - (-)^{\frac{1}{2}n} a_m^2 \}
\end{aligned}$$

expressions in which the numbers are much lower than in the case of the binomial quantic; representing the symmetric functions of $(1, a_1, \ldots \between x, 1)^n = 0$ by their partitions in ( ), we have

$$\begin{array}{rlrl} q_2 &= -(2), & q_6 &= -(2^3) \\ c_3 &= -(3), & c_7 &= -(32^2) \\ q_4 &= +(2^2), & q_8 &= +(2^4) \\ c_5 &= +(32), & c_9 &= +(32^3) \\ q_{2m} &= (-)^m(2^m); & c_{2m+1} &= (-)^m(32^{m-1}); \end{array}$$

since the partition involved for any weight is the simplest non-unitary one of the number representing the weight, we obtain conclusive evidence that the most advantageous baseforms are those of alternately two and three degrees. For weight w we have a seminvariant of extent = weight corresponding to each non-unitary partition of w, and all these are linearly independent; the degree of each is given by the highest part in the corresponding partition, or, what comes to the same thing, by the number of parts in the conjugate partition; but such a system of linearly independent seminvariants of a given weight is not the simplest that can be formed, for by adding to any seminvariant certain numerical multiples of the seminvariants of lower degrees a certain number of the terms of highest extent can be made to vanish, and we can choose the multipliers so that a maximum number shall vanish; we thus obtain another simple system of the same number of forms, and each form of a certain degree in one system will produce another of the same degree in the other, viz. there will be a one to one correspondence between the two systems. Hence arises a theorem for the number of linearly independent seminvariants of a given degree and weight, considered '*inter alia*'; thus for degree-weight (j, w) the number is equal to the number of non-unitary partitions of w which contain a part j and no part greater than j; that is, it is the coefficient of x^w in $\dfrac{x^j}{(1-x^2)(1-x^3)\ldots(1-x^j)}$, which therefore is the generating function for the number of seminvariants in question.

In the case $(j, w) = (5, 11)$ the expanded form is

$$x^5 + x^7 + x^8 + 2x^9 + 2x^{10} + 3x^{11} + \ldots$$

showing that the number is three, which we know to be the case, as they must correspond to the partitions $542 . 53^2 . 52^3$. But considering the asyzygetic seminvariants of degree-weight (5.11) '*ab alia*,' we know from Professor Cayley's rule that there are four; and since there are but three when all those of the weight 11 are considered '*inter alia*,' it follows that there must be a syzygant of degree-weight (5.11); using [ ] to denote degree-weight, we know from Mr. Hammond's late researches that

$$[2.4][3.7]-[2.6][3.5]=[1.0][4.11].$$

Syzygants of any degree-weight can therefore be detected in this manner.

Mr. Durfee (Johns Hopkins University Circulars, Dec. 1882) has shown the most advantageous way of arranging the terms of a symmetric function, viz. representing the combinations of coefficients by their partitions, the first half (not counting the self-conjugate ones) in ascending order as to number of parts, and in dictionary order in each set of so many parts; then the self-conjugate partitions; then the conjugates of the first half in inverse order.

It seems natural therefore to arrange the terms of the single-partition seminvariants according to the same plan; the ending terms then will have partitions which are the conjugates of the partitions of the symmetric functions to which they respectively correspond, and this will remain so after the asyzygetic series has been formed; consequently an ending term must have a partition, the conjugate of a non-unitary partition.

The literal expressions of the asyzygetic seminvariants up to weight 12 are annexed, the first set having reference to the quantic with derived coefficients, and the second set referring to the quantic $(a_0, a_1 \ldots \between x, y)^n$; the terms have been arranged as explained above, with the exception of the case weight 11, and a very singular symmetry becomes evident, viz. the tables are of exactly the same shape when inverted; this shows that to each seminvariant corresponds another, the partitions of whose terms are the conjugates of its own; it follows therefore that whenever there is an odd number of asyzygetic seminvariants, there must be at least one self-conjugate seminvariant; that is, one containing no term without at the same time containing another whose partition is conjugate to its own; such forms are shown by the dark vertical lines; the pairs of conjugate seminvariants are arranged symmetrically about these in the same way exactly as the pairs of terms are arranged about the self-conjugate terms; the asterisks are placed in those squares which are empty in virtue either (1) the ending term denoting the degree of the form, or (2) the leading term denoting the extent. The asterisks and blank spaces each enjoy the same symmetry.

A slightly different arrangement was found preferable in the case of weight 11; the first half of the terms are altered so as to bring those of like extent together. The leading terms are the non-unitary partitions, and the law of their correspondence with the ending terms appears to be difficult. With one exception the following law holds, viz. the m^{th} leading term from the left has for its ending term the conjugate of the m^{th} leading term from the right; the exception is in the case of the self-conjugate seminvariants of weight 11, viz. an interchange of ending terms occurs between them; if this were not so they could not both be self-conjugate.

Proceeding to develop the protomorph expressions of these seminvariants as far as weight 11, the following scheme shows for each weight: (i) the protomorph expressions of the single partition forms, (ii) the partition expressions of the asyzygetic series, (iii) the protomorph expression of the latter; the forms are numbered successively i, ii, iii . . .

WEIGHT 2.

(i) (ii) and (iii)

	q_2
No. 1. (2)	−1

WEIGHT 3.

(i) (ii) and (iii)

	c_3
No. 1. (3)	−1

WEIGHT 4.

(i)

	q_4	q_2^2
(4)	−2	+1
(2^2)	+1	

(ii)

	(2^2)	(4)
No. 1	+1	
No. 2	+2	+1

(iii)

	q_2^2	q_4
No. 1		+1
No. 2	+1	

WEIGHT 5.

(i)

	c_5	c_3q_2
(5)	−1	+1
(32)	+1	

(ii)

	(32)	(5)
No. 1	+1	
No. 2	+1	+1

(iii)

	c_3q_2	c_5
No. 1		+1
No. 2	+1	

WEIGHT 6.

	q_6	q_4q_2	c_3^2	q_2^3
(6)	-3	$+3$		-1
(42)	$+3$	-1		
$2(3^2)$	$+3$	-3	$+1$	$+1$
(2^3)	-1			

	(2^3)	(3^2)	(42)	(6)
No. 1	-1			
No. 2	-3	-2		
No. 3	$+6$	-2	$+3$	
No. 4	$+6$		$+3$	$+1$

	q_2^3	c_3^2	q_2q_4	q_6
No. 1				$+1$
No. 2	-1	-1	$+3$	
No. 3	$+1$	$+1$		
No. 4	$+1$			

Here, for the first time, a syzygant is evident, for No. 2 weight 6 is equal to

$$-3(2^3)-2(3^2)=-q_2^3-c_3^2+3q_2q_4,$$

the left-hand side of this equation is obviously of degree 3, whereas the right-hand side is of degree 6; consequently it must divide out by a_0^3, leaving a syzygant of degree 3; this is, "*mutatis mutandis*," T in Mr. Hammond's notation, a ground-source for the quartic and all higher quantics; it is convenient to write it T_{364} so that the degree, weight and rank will be denoted.

No. 3 weight 6 is $+6(2^3)-2(3^2)+3(42)=q_2^3+c_3^2$, showing that the right-hand side must divide out by a_0^2 leaving a ground syzygant for the cubic, which, "*mutatis** *mutandis*," is Δ_{463}.

But since it may also be written

$$-T_{364}+3q_2q_4,$$

we see that Δ_{463} is not a ground-source for the quartic or any higher quantic.

WEIGHT 7.

(i)

	c_7	c_5q_2	q_4c_3	$c_3q_2^2$
(7)	-1	$+1$	$+1$	-1
(52)	$+1$		-1	
(43)	$+1$	-1	$+1$	
(32^2)	-1			

(ii)

	(32^2)	(43)	(52)	(7)
No. 1	-1			
No. 2	-1	-1		
No. 3	-1		-1	
No. 4	-2	-1	-2	-1

(iii)

	$c_3q_2^2$	q_4c_3	c_5q_2	c_7
No. 1				$+1$
No. 2		-1	$+1$	
No. 3		$+1$		
No. 4	$+1$			

No. 2 weight 7 is $-(32^2)-(43)=-q_4c_3+c_5q_2$, showing that a_0 must be a factor, leaving P_{475} a ground-source for the quintic and all higher quantics.

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* This will be understood in future.

Weight 8.

(i)

	q_8	q_6q_2	c_5c_3	q_4^2	$q_4q_2^2$	$c_3^2q_2$	q_2^4
(8)	−4	+4		+2	−4		+1
(62)	+4	−1		−2	+1		
(53)	+4	−4	+1	−2	+4	−1	−1
(4^2)	+2	−2		+1			
(42^2)	−4	+1					
$2(3^22)$	−8	+5	−2	+4	−5	+1	+1
(2^4)	+1						

(ii)

	(2^4)	(3^22)	(42^2)	(4^2)	(53)	(62)	(8)
No. 1	+1						
No. 2	+8	+2					
No. 3	+12	−2	+5				
No. 4	+6		+2	+1			
No. 5	+12	+2	+3		+2		
No. 6	−12	−2			−2	+3	
No. 7	+48		+20	+10		+4	+1

(iii)

	q_2^4	$c_3^2q_2$	$q_4q_2^2$	q_4^2	c_5c_3	q_6q_2	q_8
No. 1							+1
No. 2	+1	+1	−5	+4	−2	+5	
No. 3	−1	−1	+5	−4	+2		
No. 4				+1			
No. 5	−1	−1	+3				
No. 6	+1	+1					
No. 7	+1						

No. 2 weight 8 has evidently a factor a_0^5 and is a syzygant, say K_{386}, a ground-source for the sextic and all higher quantics.

No. 3 weight 8 has a factor a_0^4 and is a ground-source, say L_{485}, deg. ord. (4.4) for the quintic, but since it may be expressed in the form $-K_{386}+5q_6q_2$ it is not a ground-source for the sextic and higher quantics.

Nos. 5 and 6 reduce respectively to q_2T_{364} and $q_2\Delta_{463}$. The analysis of weight 8 therefore is complete.

Weight 9.

(i)

	c_9	c_7q_2	q_6c_3	c_5q_4	$c_5q_2^2$	$q_4c_3q_2$	c_3^3	$c_3q_2^3$
(9)	-1	+1	+1	+1	-1	-2		+1
(72)	+1		-1	-1		+1		
(63)	+1	-1	+2	-1	+1	-1		
(54)	+1	-1	-1	+1				
(52^2)	-1		+1					
(432)	-2	+1	-1					
(3^3)	-2	+2	-7	+2	-2	+5	-1	-1
(32^3)	+1							

(ii)

	(32^3)	(3^3)	(432)	(52^2)	(54)	(63)	(72)	(9)
No. 1	+1							
No. 2	+2	+6						
No. 3	+2	-6	+2					
No. 4	+3	+1	-2	+5				
No. 5	+3	-1	+4	-1	+2			
No. 6	+3	+6		+3		+2		
No. 7	+6	-6	+3	+6	+3	-2	+3	
No. 8	+6		+3	+6	+3	+1	+3	+1

(iii)

	$c_3q_2^3$	c_3^3	$q_4c_3q_2$	$c_5q_2^2$	c_5q_4	q_6c_3	c_7q_2	c_9
No. 1								+1
No. 2	-1	-1	+5	-2	+2	-7	+2	
No. 3	+1	+1	-5	+2	-2	+5		
No. 4	-1	-1	+5	-2	+2			
No. 5	+1	+1	-5	+2				
No. 6	-1	-1	+3					
No. 7	+1	+1						
No. 8	+1							

No. 2 weight 9 is obviously divisible by a_0^6, and is therefore a syzygant giving R_{397} a ground-source for the septimic and all higher quantics.

No. 3 is, after rejection of the factor a_0^5, S_{496} a ground-source for the sextic and all higher quantics.

No. 5 is a ground-source for the quintic Λ_{595}, a_0^4 being the integrating factor, but since it can be written

$$2c_5 q_4 - 10 q_6 c_3 + 2S_{496},$$

it is not so for the sextic and higher quantics.

No. 4 is now seen to be $2c_5 q_4 - \Lambda_{595}$, which is of the right degree.

No. 6, of degree 6, is $c_3 T_{364}$.

No. 7, of degree 7, is $c_3 \Delta_{463}$.

WEIGHT 10.

(i)

	q_{10}	$q_8 q_2$	$c_7 c_3$	$q_6 q_4$	$q_6 q_2^2$	c_5^2	$c_5 c_3 q_2$	$q_4^2 q_2$	$q_4 c_3^2$	$q_4 q_2^3$	$c_3^2 q_2^2$	q_2^5
(10)	−5	+5		+5	−5			−5		+5		−1
(82)	+5	−1		−5	+1			+3		−1		
(73)	+5	−5	+1	−5	+5		−1	+5	−1	−5	+1	+1
(64)	+5	−5		+1	+2			−1				
(62^2)	−5	+1		+2	−1							
$2(5^2)$	+5	−5		−5	+5	+1	−2	+5		−5	+1	+1
(532)	−10	+6	−1	+10	−6	−1	+2	−8	+1	+6	−1	−1
$(4^2 2)$	−5	+3		−1								
$2(43^2)$	−10	+10	−2	+4	−7		+2	−4		+5	−1	−1
(42^3)	+5	−1										
$2(3^2 2^2)$	+15	−7	+2	−12	+7	+1	−2	+8	−1	−6	+1	+1
(2^5)	−1											

(ii)

	(2^5)	$(3^2 2^2)$	(42^3)	(43^2)	$(4^2 2)$	(532)	(5^2)	(62^2)	(64)	(73)	(82)	(10)
No. 1	−1											
No. 2	−15	−2										
No. 3	−20	+2	−7									
No. 4	−20	−2	−3	−2								
No. 5	−60	+2	−21	+2	−8							
No. 6	−40	+4	−8	−2		−2						
No. 7	−30	−2	−9		−3	−2	−2					
No. 8	−60	+4	−27	+2	−10	+2		−5				
No. 9	−60	+2	−27		−12	+2	+2	−6	−3			
No. 10	−60	−2	−21	−1	−6	−4	−2	−3		−2		
No. 11	−120	+2	−60	+1	−30	+4	+2	−18	−9	+2	−3	
No. 12	+120		+60		+30			+20	+10		+5	+1

(iii)

	q_2^5	$c_3^2q_2^2$	$q_4q_2^3$	$q_4c_3^2$	$q_4^2q_2$	$c_5c_3q_2$	c_5^2	$q_6q_2^2$	q_6q_4	c_7c_3	q_8q_2	q_{10}
No. 1												+1
No. 2	−1	−1	+6	+1	−8	+2	−1	−7	+12	−2	+7	
No. 3	+1	+1	−6	−1	+8	−2	+1	+7	−12	+2		
No. 4			+2	+2	−8		−2		+16			
No. 5			−1	−1	+4		+1					
No. 6	+1	+1	−5		+4	−2		+5				
No. 7			−1	−1	+3							
No. 8	−1	−1	+5		−4	+2						
No. 9			+1	+1								
No. 10	−1	−1	+3									
No. 11	+1	+1										
No. 12	+1											

No. 2 weight 10 has a factor a_0^7 and is a ground-source deg. ord. 3.4 for the octavic and all higher quantics; say $S_{3,10,8}$.

No. 3 has a factor a_0^6, leaving a ground-source deg. ord. 4.8 for the septimic, but since it may obviously be written $7q_8q_2 - S_{3,10,8}$ it is not an irreducible form for the octavic and higher quantics.

Nos. 4 and 5 are not really separate forms; the latter is divisible by a_0^4, leaving a ground-source deg. ord. 4.0 for the quintic and all higher quantics.

No. 6 is simply $q_2 K_{386}$.

No. 7 is equal to $q_4 T_{364}$.

No. 8 " " $q_2 L_{485}$.

No. 9 " " $q_4 \Delta_{463}$.

No. 10 " " $q_2^2 T_{364}$.

No. 11 " " $q_2^2 \Delta_{463}$, so that there exist no more ground-sources of weight 10.

Weight 11.

(i)

	c_{11}	c_9q_2	q_8c_3	c_7q_4	$c_7q_2^2$	q_6c_5	$q_6c_3q_2$	$c_5q_4q_2$	$c_5c_3^2$	$c_5q_2^3$	$q_4^2c_3$	$q_4c_3q_2^2$	$c_3^3q_2$	$c_3q_2^4$
(11)	−1	+1	+1	+1	−1	+1	−2	−2		+1	−1	+3		−1
(92)	+1		−1	−1		−1	+1	+1			+1	−1		
(83)	+1	−1	+3	−1	+1	−1	−2	+2		−1	−1	+1		
(74)	+1	−1	−1	+1		−1	+2				−1			
(72^2)	−1		+1			+1	−1							
(65)	+1	−1	−1	−1	+1	+2	−1	−1			+1			
(632)	−2	+1	−2	+2	−1	−1	+1							
(542)	−2	+1	+2			−1								
$2(53^2)$	−2	+2	−6	+2	−2	−1	+7	−1	−1	+1	+2	−5	+1	+1
(52^3)	+1		−1											
(4^23)	−1	+1	−1	−1		+1								
(432^2)	+3	−1	+1											
$6(3^32)$	+6	−4	+18	−6	+4	+3	−14	+1	+3	−1	−6	+10	−2	−2
(32^4)	−1													

(ii)

	(32^4)	(3^32)	(432^2)	(4^23)	(52^3)	(53^2)	(542)	(632)	(65)	(72^2)	(74)	(83)	(92)	(11)
No. 1	—1													
No. 2	—6	—6												
No. 3	—6	+6	—4											
No. 4	—2		—1	—1										
No. 5	—4	+6	—10	—6	+14									
No. 6	—4	—6	+2	+2	—6	—2								
No. 7		—6	+3	+2	—3	—4	+1							
No. 8	—8	—6			—8	—2		—2						
No. 9		+6	—6	—3	+6	+4	—3	—1	—1					
No. 10		—6				—6				+2				
No. 11	—6		—2	—1	—6		—2			—2	—1			
No. 12	—12	—6	—3		—12	—6	—3	—2	—2	—3		—2		
No. 13	—24	+6	—12	—6	—24	+6	—12	—1	—1	—12	—6	+2	—3	
No. 14	—24		—12	—6	—24		—12	—4	—4	—12	—6	—1	—4	—1

(iii)

	$c_3q_2^4$	$c_3^3q_2$	$q_4c_3q_2^2$	$q_4^2c_3$	$c_5q_2^3$	$c_5c_3^2$	$c_5q_4q_2$	$q_6c_3q_2$	q_6c_5	$c_7q_2^2$	c_7q_4	q_8c_3	c_9q_2	c_{11}
No. 1														+1
No. 2	+2	+2	−10	+6	+1	−3	−1	+14	−3	−4	+6	−18	+4	
No. 3	−2	−2	+10	−6	−1	+3	+1	−14	+3	+4	−6	+14		
No. 4									−1		+1			
No. 5	−2	−2	+10	−6	−1	+3	+1	−14	−3	+4				
No. 6				+2	−1	−1	+1		+1					
No. 7				+2	−1	−1	+1							
No. 8	+1	+1	−5	+4		−2		+5						
No. 9				−3	+1	+1								
No. 10	+1	+1	−5	−2	+2									
No. 11				+1										
No. 12	−1	−1	+3											
No. 13	+1	+1												
No. 14	+1													

No. 2 clearly has a factor a_0^8, leaving, say $M_{3,11,9}$ a ground-source for the nonic and all higher quantics; this source is deg. ord. $3.3n - 22$ for the n^{ic}.

No. 3 is $-a_0^8M_{3,11,9} + 4(c_9q_2 - q_8c_3)$, and since the expression in ( ) has a factor a_0, it is a ground-source for the octavic, say $N_{4,11,8}$, but not so for the nonic or any higher quantic; it is deg. ord. 4.10 for the octavic.

No. 4 is $c_7q_4 - q_6c_5$, the partition expression showing that it must have a factor a_0; thus, No. 4 $= c_7q_4 - q_6c_5 = a_0F_{4,11,7}$ a ground-source deg. ord. 4.6 for the septimic and all higher quantics; which is Mr. Hammond's result (*vide* Amer. Jour., Vol. V. No. 1).

Nos. 5, 6 and 7 are not separate forms. No. 7 is obviously divisible by a_0^4, and being the simplest of the three, we take as the source of the covariant deg. ord. 5.3 for the quintic, say $\Theta_{5,11,5}$.

Then No. $6 = \Theta_{5,11,5} + q_6 c_5$.

No. $5 = 2q_2 R_{397} - 3$(No. 6),

and consequently Nos. 5 and 6 are not ground-sources.

Since, however, we find easily

$$3\Theta = 3q_6 c_5 - 6c_7 q_4 + 14q_8 c_3 + 2q_2 R_{397} - N_{4,11,8}$$

Θ is seen to be expressible in terms of lower groundforms in the case of the octavic; consequently Θ is a ground-source for the sextic and septimic as well as for the quintic, but not for any higher quantic.

No. 8 is equal to $c_3 K_{386}$.
No. 9 " " $3q_4 P_{475} - c_5 T_3$.
No. 10 " " $c_3 L_{485} + 2c_5 \Delta_{463} + 6q_4^2 c_3$.
No. 12 " " $c_3 q_2 T_{364}$.
No. 13 " " $c_3 q_2 \Delta_{463}$,

which completes the consideration of weight 11.

In the calculation of single-partition seminvariants in terms of protomorphs, Mr. Hammond's other operators (*vide* Proceedings London Math. Soc., February, 1883) may be utilized; these are included in the formula $d_\lambda = [\lambda]$; where $[\lambda]$ stands for the sum of the λ^{th} powers of the roots of the equation

$$x^n + D_1 x^{n-1} + D_2 x^{n-2} + D_3 x^{n-3} + \ldots = 0,$$

and the law of operation of D on ( ) is from his paper

$$D_\kappa(\lambda^l.\mu^m.\nu^n\ldots) = 0,\quad D_\lambda(\lambda^l.\mu^m.\nu^n\ldots) = (\lambda^{l-1}.\mu^m.\nu^n\ldots),\quad D_\lambda(\lambda) = 1.$$

To calculate thus seminvariant $(3^2 2)$, put

$$(3^2 2) = \alpha_1(2^4) + \alpha_2(2^3)(2) + \alpha_3(32)(3) + \alpha_4(2^2)^2 + \alpha_5(2^2)(2)^2 + \alpha_6(3)^2(2) + \alpha_7(2)^4.$$

Operating with d_8 or $\ldots D_2^4 - 4D_3^2 D_2 \ldots$, we find at once $\alpha_1 = -4$, and the lower operators quickly give the other coefficients.

When performed on the literal expression,

$$d_\lambda = \partial_{a_\lambda} + a_1 \partial_{a_{\lambda+1}} + a_2 \partial_{a_{\lambda+2}} + \text{etc.},$$

and clearly any of these operators when performed on a seminvariant will yield seminvariants; the operation of d_3 alone frequently enables us to determine the degree in the protomorphs of any given seminvariant; if ϕ be any form and we find $d_3\phi = 0$, it shows us that ϕ consists entirely of even-weight protomorphs, since d_3 would not make a term containing a 'c' vanish; generally if m

operations of d_3 reduce a seminvariant to zero, its expression in protomorphs can contain no term with more than $m-1$ odd-weight protomorphs, and must contain a term with that number; if m operations reduce the form to a constant, then the term containing most c's contains m of them and no q; operating with d_2 and d_3 will thus disclose the degree in the protomorphs, which is the end in view.

Putting $a_0 = 0$ in any seminvariant we obtain what Professor Sylvester calls its residue, and he has explained its semi-invariantive character; in the present case the residue is a seminvariant, quâ the elements $a_1, a_2, a_3, a_4 \ldots$ that is of the quantic

$$a_1 x^n + n a_2 x^{n-1} + n(n-1) a_3 x^{n-2} + \ldots$$

since then the operator becomes

$$\partial_{a_2} + a_2 \partial_{a_3} + a_3 \partial_{a_4} + \ldots$$

which is of the same character as

$$\partial_{a_1} + a_1 \partial_{a_2} + a_2 \partial_{a_3} + \ldots$$

and differs from it merely in each suffix being increased by unity.

Putting further $a_1, a_2, a_3 \ldots a_{m-1}$, each equal to zero, we obtain the m^{th} residue which is a seminvariant of the quantic

$$a_m x^n + n a_{m+1} x^{n-1} + n(n-1) a_{m+2} x^{n-2} + \ldots$$

It follows that if $\phi(a_0, a_1, a_2 \ldots)$ be a seminvariant, $\phi(a_1, a_2, a_3 \ldots)$ is a residue, the latter being obtained from the former by increasing each suffix by unity; the protomorph residues are therefore at once written down, viz. they are

$$2a_3 a_1 - a_2^2$$
$$3a_4 a_1^2 - 3a_3 a_2 a_1 + a_2^3$$
$$2a_5 a_1 - 2a_4 a_2 + a_3^2, \text{ etc.}$$

This simple derivation of the residue groundforms is of some advantage in the exhaustive method of groundform deduction initiated by Professor Cayley; considering the residues quâ the operator

$$a_1 \partial_{a_2} + a_2 \partial_{a_3} + a_3 \partial_{a_4} + \ldots$$

the residue syzygants will contain some power of a_1 as a factor; supposing the partitions in $\{\quad\}$ to refer to the seminvariants of the quantic

$$a_1 x^n + n a_2 x^{n-1} + n(n-1) a_3 x^{n-2} + \ldots$$

and $\mathfrak{R}$ to denote the residue of the seminvariant which follows it, then

$$\mathfrak{R}(2^\nu) = \{2^{\nu-1}\};\ \mathfrak{R}(32^\nu) = -a_1\{2^\nu\};\ \mathfrak{R}(2) = a_1^2,$$

and generally

$$\mathfrak{R}(\lambda^l \mu^m \nu^n \ldots) = (-1)^\lambda \{\lambda^{l-1} \mu^m \nu^n \ldots\},$$
$$\mathfrak{R}(\lambda \mu^m \nu^n \ldots) = (-1)^\lambda a_1^{\lambda-\mu} \{\mu^m \nu^n \ldots\};$$

consequently, if in the protomorph expression of any seminvariant there be either an odd weight or a weight two (q_2) protomorph in each term, a_1 will be a factor of the residue; generally if i, t denote respectively the number of odd weight and weight two protomorphs in a term, and the least value of $i + 2t$ that occurs in every term be ν, then a_1^ν is a factor of the residue, and conversely if a_1^ν is a factor of the residue, $j + 2t$ must be at least equal to ν in every term of the protomorph expression.

From the facts above given it will be seen that starting from a seminvariant of degree weight (j, w), we can immediately write down the residue of a seminvariant of degree-weight $(j, w+j)$; if the former is a member of the weight w asyzygetic series, the latter will not necessarily be a member of the weight $w+j$ series, but, more often than not, will be a linear combination of the members of that series; on this principle the ending terms of a weight w series can be derived from those of the several series of inferior weight in regular succession. Starting from the ending terms of zero weight

$$a_0,\ a_0^2,\ a_0^3,\ a_0^4,\ \ldots$$

by a unit increase of suffixes we derive ending terms $a_1^2, a_1^3, a_1^4, \ldots$ of weights $2, 3, 4 \ldots$; the ending terms of weight 2 are therefore a_1^2, $a_0a_1^2$, $a_0^2a_1^2, \ldots$ and from these we derive ending terms a_2^2, $a_1a_2^2$, $a_1^2a_2^2, \ldots$ of weights 4, 5, 6; from the ending terms of weight 3, a_1^3, $a_0a_1^3$, $a_0^2a_1^3, \ldots$ we derive a_2^3, $a_1a_2^3$, $a_1^2a_2^3, \ldots$ of weights $6, 7, 8 \ldots$; thus the ending terms of weight 4 are a_1^4 and a_2^2, and of weight 5 a_1^5 and $a_1a_2^2$, and so forth; we can thus not only derive the ending terms of higher weights, but also, to numerical factors près, the residues of certain seminvariants having those ending terms. Ex. gr.

Suppose we wish to derive the residues of the weight 8 from the seminvariants of lower weight; we know that the ending terms of weights 0, 2, 3, 4, 5, 6, 7, are respectively

$$(a_0^*)(a_1^2)(a_1^3)(a_1^4,\ a_2^2)(a_1^5,\ a_1a_2^2)(a_1^6,\ a_1^2a_2^2,\ a_2^3,\ a_3^2)(a_1^7,\ a_1^3a_2^2,\ a_1a_2^3,\ a_1a_3^2)\,;$$

whence we derive by a unit increase of suffixes,

From $a_0^8 \ldots a_1^8$
" $a_0^4a_1^2 \ldots a_1^4a_2^2$
" $a_0^2a_1^3 \ldots a_1^2a_2^3$
" $a_1^4 \ldots a_2^4$
" $a_0^2a_2^2 \ldots a_1^2a_3^2$
" $a_1a_2^2 \ldots a_2a_3^2$
" $a_3^2 \ldots a_4^2$

no other derivation being possible; in fact, if (j, w) be the degree-weight of a lower ending term when multiplied by any positive integer including zero power of a_0, then we must have $$j+w=8;$$
the above are the only possible solutions of this equation; so that if we multiply a seminvariant of lower weight by such a power of a_0 that, as regards the ending term: $$j+w=8,$$
a unit increase of suffixes will give, to a numerical factor pres, the residue of a seminvariant of weight 8 and degree j; if we wish to form the residue of a seminvariant of weight w_2 from a form deg-weight (w_1, j_1), we multiply the latter by $a_0^{w_2-w_1-j_1}$ where $w_2-w_1-j_1$ is not less than zero, and then a unit increase of suffixes gives, to a numerical factor pres, the required residue of a seminvariant of degree-weight (w_2-w_1, w_2); so that the number of forms of degree-weight (j_2, w_2) in the asyzygetic series must be the equal to the number of forms of weight w_2 that it is possible to derive from the weight (w_2-j_2) asyzygetic series; this is clearly equal to the number of non-unitary partitions of w_2-j_2 which contain no part greater than j_2; this result is obviously identical with that obtained at the beginning of the paper, viz. that the number is equal to the number of non-unitary partitions of w_2 which contain a part j_2 and no part greater than j_2.

Since the operator $a_0\partial_{a_1}+2a_1\partial_{a_2}+3a_2\partial_{a_3}+\ldots$
$$=(a_0\partial_{a_1}+a_1\partial_{a_2}+a_2\partial_{a_3}+\ldots)+(a_1\partial_{a_2}+a_2\partial_{a_3}+a_3\partial_{a_4}+\ldots)$$
$$+(a_2\partial_{a_3}+a_3\partial_{a_4}+a_4\partial_{a_5}+\ldots)+\ldots$$
we see that the seminvariants of the quantic
$$(a_0, a_1, a_2 \ldots \between x, y)^n,$$
can be expressed in terms of those of the quantics whose type is
$$a_m x^n+na_{m+1}x^{n-1}+n(n-1)a_{m+2}x^{n-2}+\ldots$$

ASYZYGETIC SEMINVARIANTS.

Tables A *for Quantics with Binomial Coefficients; tables* B *for ditto with Derived Coefficients.*

TABLES 1A and 1B.

a_0	+1

TABLE 2A.

a_2	+1
a_1^2	−1

TABLE 2B.

a_2	+2
a_1^2	−1

TABLE 3A.

a_3	+1
a_1a_2	−3
a_1^3	+2

TABLE 3B.

a_3	+3
a_1a_2	−3
a_1^3	+1

TABLE 4A.

a_4	+1	
a_1a_3	−4	
a_2^2	+3	+1
$a_1^2a_2$		−2
a_1^4		+1

TABLE 4B.

a_4	+2	
a_1a_3	−2	
a_2^2	+1	+4
$a_1^2a_2$		−4
a_1^4		+1

TABLE 5A.

a_5	+1	
a_1a_4	−5	
a_2a_3	+2	+1
$a_1^2a_3$	+8	−1
$a_1a_2^2$	−6	−3
$a_1^3a_2$		+5
a_1^5		−2

TABLE 5B.

a_5	+5	
a_1a_4	−5	
a_2a_3	+1	+6
$a_1^2a_3$	+2	−3
$a_1a_2^2$	−1	−6
$a_1^3a_2$		+5
a_1^5		−1

Table 6A.

a_6	+1			
a_1a_5	—6			
a_2a_4	+15	+1		
a_3^2	—10	—1	+1	
$a_1^2a_4$		—1	*	
$a_1a_2a_3$		+2	—6	
$a_1^3a_3$		*	+4	
a_2^3		—1	+4	+1
$a_1^2a_2^2$			—3	—3
$a_1^4a_2$				+3
a_1^6				—1

Table 6B.

a_6	+2			
a_1a_5	—2			
a_2a_4	+2	+12		
a_3^2	—1	—9	+9	
$a_1^2a_4$		—6	*	
$a_1a_2a_3$		+6	—18	
$a_1^3a_3$		*	+6	
a_2^3		—2	+8	+8
$a_1^2a_2^2$			—3	—12
$a_1^4a_2$				+6
a_1^6				—1

Table 7A.

a_7	+1			
a_1a_6	—7			
a_2a_5	+9	+1		
a_3a_4	—5	—1	+1	
$a_1^2a_5$	+12	—1	*	
$a_1a_2a_4$	—30	—2	—3	
$a_1a_3^2$	+20	+4	—4	
$a_1^3a_4$		+3	+2	
$a_2^2a_3$		—1	+3	+1
$a_1^2a_2a_3$		—6	+12	—2
$a_1^4a_3$		*	—8	+1
$a_1a_2^3$		+3	—9	—3
$a_1^3a_2^2$			+6	+8
$a_1^5a_2$				—7
a_1^7				+2

Table 7B.

a_7	+7			
a_1a_6	—7			
a_2a_5	+3	+10		
a_3a_4	—1	—6	+6	
$a_1^2a_5$	+2	—5	*	
$a_1a_2a_4$	—2	—4	—6	
$a_1a_3^2$	+1	+6	—6	
$a_1^3a_4$		+3	+2	
$a_2^2a_3$		—1	+3	+12
$a_1^2a_2a_3$		—3	+6	—12
$a_1^4a_3$		*	—2	+3
$a_1a_2^3$		+1	—3	—12
$a_1^3a_2^2$			+1	+16
$a_1^5a_2$				—7
a_1^7				+1

TABLE 8A.

a_8	+1						
a_1a_7	−8						
a_2a_6	+28	+1					
a_3a_5	−56	−3	+3				
a_4^2	+35	+2	−2	+1			
$a_1^2a_6$		−1	*	*			
$a_1a_2a_5$		+3	−9	*			
$a_1a_3a_4$		−1	+1	−8			
$a_2^2a_4$		−3	+18	+6	+1		
$a_1^3a_5$		*	+6	*	*		
$a_2a_3^2$		+2	−12				
$a_1^2a_2a_4$			−15		−2		
$a_2a_3^2$					−1	+1	
$a_1^4a_4$			*	*	+1	*	
$a_1^2a_3^2$			+10	+16	+1	−1	
$a_1a_2^2a_3$				−24	+2	−6	
$a_1^3a_2a_3$				*	−2	+10	
$a_1^5a_3$				*	*	−4	
a_2^4				+9	−1	+4	+1
$a_1^2a_2^3$					+1	−7	−4
$a_1^4a_2^2$						+3	+6
$a_1^6a_2$							−4
a_1^8							+1

TABLE 8B.

a_8	+2						
a_1a_7	−2						
a_2a_6	+2	+10					
a_3a_5	−2	−15	+30				
a_4^2	+1	+8	−16	+4			
$a_1^2a_6$		−5	*	*			
$a_1a_2a_5$		+5	−30	*			
$a_1a_3a_4$		−1	+2	−8			
$a_2^2a_4$		−2	+24	+4	+24		
$a_1^3a_5$		*	+10	*	*		
$a_2a_3^2$		+1	−12				
$a_1^2a_2a_4$			−10		−24		
$a_2a_3^2$					−18	+18	
$a_1^4a_4$			*	*	+6	*	
$a_1^2a_3^2$			+5	+4	+9	−9	
$a_1a_2^2a_3$				−4	+12	−36	
$a_1^3a_2a_3$				*	−16	+30	
$a_1^5a_3$				*	*	−6	
a_2^4				+1	−4	+16	+16
$a_1^2a_2^3$					−2	−14	−32
$a_1^4a_2^2$						+3	+24
$a_1^6a_2$							−8
a_1^8							+1

TABLE 9A.

a_9	+1							
a_1a_8	−9							
a_2a_7	+20	+2						
a_3a_6	−28	−7	+1					
a_4a_5	+14	+5	−1	+1				
$a_1^2a_7$	+16	−2	*	*				
$a_1a_2a_6$	−56	+7	−3	*				
$a_1a_3a_5$	+112	+22	−2	−4				
$a_2^2a_5$		−27	+9	−9	+2			
$a_1a_4^2$	−70	−25	+5	−5				
$a_2a_3a_4$		+45	−17	+32	−5	+1		
$a_1^3a_6$		*	+2	*	*	*		
$a_1^2a_2a_5$		*	−6	+24	−4	*		
$a_1^2a_3a_4$		*	+2	−2	+5	−1		
$a_1^4a_5$		*	*		+2	*		
a_3^3		−20	+8	−18	+3	−1	+1	
$a_1^4a_5$			*	−12		*	*	
$a_1a_2^2a_4$			+6	−51	+5	−3	*	
$a_1^3a_2a_4$			*	+30	−5	+5	*	
$a_1^5a_4$			*	*	*	−2	*	
$a_1a_2a_3^2$			−4	+34	−7	+5	−9	
$a_2^3a_3$					+1	−1	+4	+1
$a_1^3a_3^2$				−20	−2	−2	+6	
$a_1^2a_2^2a_3$					+8	−6	+15	−3
$a_1^4a_2a_3$					*	+4	−24	+3
$a_1^6a_3$					*	*	+8	−1
$a_1a_2^4$					−3	+3	−12	−3
$a_1^3a_2^3$						−2	+17	+11
$a_1^5a_2^2$							−6	−15
$a_1^7a_2$								+9
a_1^9								−2

TABLE 9B.

a_9	+9							
a_1a_8	−9							
a_2a_7	+5	+14						
a_3a_6	−3	−21	+30					
a_4a_5	+1	+10	−20	+20				
$a_1^2a_7$	+2	−7	*	*				
$a_1a_2a_6$	−2	+7	−30	*				
$a_1a_3a_5$	+2	+11	−5	−20				
$a_2^2a_5$		−9	+30	−15	+40			
$a_1a_4^2$	−1	−10	+20	−20				
$a_2a_3a_4$		+9	−34	+64	−60	+36		
$a_1^3a_6$		*	+10	*	*	*		
$a_1^2a_2a_5$		*	−10	+40	−40	*		
$a_1^2a_3a_4$		*	+2	−10	+30	−18		
$a_1^4a_5$		*	*		+10	*		
a_3^3		−3	+12	−27	+27	−27	+27	
$a_1^4a_5$			*	−10		*	*	
$a_1a_2^2a_4$			+4	−34	+20	−36	*	
$a_1^3a_2a_4$			*	+10	−10	+50	*	
$a_1^5a_4$			*	*	*	−6	*	
$a_1a_2a_3^2$			−2	+17	−21	+45	−51	
$a_2^3a_3$					+2	−6	+24	+24
$a_1^3a_3^2$				−5	−3	−9	+27	
$a_1^2a_2^2a_3$					+8	−6	+45	−36
$a_1^4a_2a_3$					*	+2	−36	+18
$a_1^6a_3$					*	*	+6	−3
$a_1a_2^4$					−2	+6	−24	−24
$a_1^3a_2^3$						−2	+17	+44
$a_1^5a_2^2$							−3	−30
$a_1^7a_2$								+9
a_1^9								−1

TABLE 10A.

a_{10}	+1											
a_1a_9	−10											
a_2a_8	+45	+1										
a_3a_7	−120	−4	+4									
a_4a_6	+210	+8	−8	+16								
a_5^2	−126	−5	+5	−15	+1							
$a_1^2a_8$		−1	*	*	*							
$a_1a_2a_7$		+4	−12	*	*							
$a_1a_3a_6$		−4	+4	−64	*							
$a_2^2a_6$		−4	+32	+48	*	+1						
$a_1a_4a_5$		+2	−2	+54	−10							
$a_2a_3a_5$		+8	−64	−60	+4	−3	+3					
$a_2a_4^2$		−5	+40		+16	+2	−2	+1				
$a_3^2a_4$				+20	−12			−1	+1			
$a_1^3a_7$			+8	*	*	*	*	*	*			
$a_1^2a_2a_6$			−28		*	−2	*	*	*			
$a_1^2a_3a_5$			+56	+144	+16	+3	−3	*	*			
$a_1a_2^2a_5$				−108	−12	+3	−9	*	*			
$a_1^2a_4^2$			−35	−135	+9	−2	+2	−1				
$a_1^4a_6$				*	*	+1	*	*	*			
$a_1^3a_2a_5$				*	*	−3		*	*			
$a_1a_2a_3a_4$				+180	−76	−1	+1	−2	−6			
$a_1^3a_2a_5$				*	*		+15	*	*			
$a_1^5a_5$				*	*	*	−6	*	*			
$a_2^3a_4$					+48	−3	+18	+2	+4	+1		
$a_1^3a_3a_4$				*	*	+1	−1	+4	+4			
$a_1^2a_2^2a_4$				*	*	+3	−33	−3	−3	−3		
$a_1^4a_2a_4$				*	*	*	+15	*		+3		
$a_1^6a_4$				*	*	*	*	*	*	−1		
$a_1a_3^3$				−80	+48			+4	−4			
$a_2^2a_3^2$					−32	+2	−12	−3	+3	−1	+1	
$a_1^2a_2a_3^2$						−2	+22	−8	+24	+2	−2	
$a_1a_2^3a_3$								+10	−34	+2	−6	
$a_1^4a_3^2$							−10	*	−16	−1	+1	
$a_1^3a_2^2a_3$								*	+24	−4	+16	
$a_1^5a_2a_3$								*	*	+2	−14	
$a_1^7a_3$								*	*	*	+4	
a_2^5								−3	+12	−1	+4	+1
$a_1^2a_2^4$									−9	+2	−11	−5
$a_1^4a_2^3$										−1	+10	+10
$a_1^6a_2^2$											−3	−10
$a_1^8a_2$												+5
a_1^{10}												−1

TABLE 10B.

a_{10}	+2											
a_1a_9	—2											
a_2a_8	+2	+28										
a_3a_7	—2	—42	+42									
a_4a_6	+2	+48	—48	+32								
a_5^2	—1	—25	+25	—25	+25							
$a_1^2a_8$		—14	*	*	*							
$a_1a_2a_7$		+14	—42	*	*							
$a_1a_3a_6$		—6	+6	—32	*							
$a_2^2a_6$		—4	+32	+16	*	+20						
$a_1a_4a_5$		+2	—2	+18	—50							
$a_2a_3a_5$		+4	—32	—10	+10	—30	+60					
$a_2a_4^2$		—2	+16		+32	+16	—32	+24				
$a_3^2a_4$				+2	+18			+18	+18			
$a_1^3a_7$			+14	*	*	*	*	*	*			
$a_1^2a_2a_6$			—14		*	—20	*	*	*			
$a_1^2a_3a_5$			+14	+12	+20	+15	—30	*	*			
$a_1a_2^2a_5$				—6	—10	+10	—60	*	*			
$a_1^2a_4^2$			—7	—9	+9	—8	+16	—12				
$a_1^4a_6$				*	*	+5	*	*	*			
$a_1^3a_2a_5$				*	*	—5		*	*			
$a_1a_2a_3a_4$				+6	—38	—2	+4	—12	—36			
$a_1^3a_2a_5$				*	*		+50	*	*			
$a_1^5a_5$				*	*	*	—10	*	*			
$a_2^3a_4$					+16	—4	+48	+8	+16	+48		
$a_1^3a_3a_4$				*	*	+1	—2	+12	+12			
$a_1^2a_2^2a_4$				*	*	+2	—44	—6	—6	—72		
$a_1^4a_2a_4$				*	*	*	+10	*		+36		
$a_1^6a_4$				*	*	*	*	*	*	—6		
$a_1a_3^3$				+2	+18			+18	—18			
$a_2^2a_3^2$					—8	+2	—24	—9	+9	—36	+36	
$a_1^2a_2a_3^2$						—1	+22	—12	+36	+36	—36	
$a_1a_2^3a_3$								+10	—34	+24	—72	
$a_1^4a_3^2$							—5	*	—12	—9	+9	
$a_1^3a_2^2a_3$								*	+12	—24	+96	
$a_1^5a_2a_3$								*	*	+6	—42	
$a_1^7a_3$								*	*	*	+6	
a_2^5								—2	+8	—8	+32	+32
$a_1^2a_2^4$									—3	+8	—44	—80
$a_1^4a_2^3$										—2	+20	+80
$a_1^6a_2^2$											—3	—40
$a_1^8a_2$												+10
a_1^{10}												—1

TABLE 11A.

a_{11}	+1											
a_1a_{10}	−11											
a_2a_9	+35	+2										
$a_1^2a_9$	+20	−2										
a_3a_8	−75	−9	+1									
$a_1a_2a_8$	−90	+9	−3									
$a_1^3a_8$	*	*	+2									
a_4a_7	+90	+14	−2	+1								
$a_1a_3a_7$	+240	+16		−4								
$a_2^2a_7$		−30	+10	+3	+16							
$a_1^2a_2a_7$	*	*	−8		−32							
$a_1^4a_7$	*	*	*	*	+16							
a_5a_6	−42	−7	+1	−1	−1	+1						
$a_1a_4a_6$	−420	−63	+9	−2	+5	−5						
$a_2a_3a_6$		+70	−26	−2	−58	+2	+1					
$a_1^2a_3a_6$	*	*	+8	+20	+48	+8	−1					
$a_1a_2^2a_6$	*	*	+8	−15	+62	−6	−3					
$a_1^3a_2a_6$	*	*	*	*	−56		+5					
$a_1a_5^2$	+252	+42	−6	+6	+6	−6						
$a_2a_4a_5$		−21	+7	−6	−5	+45		+1				
$a_1^2a_4a_5$		*	−4	−18	−40			−1				
$a_3^2a_5$		−56	+24	+10	+100	−100	−3	−3	+1			
$a_1a_2a_3a_5$		*	−16	−24	−232	+408	+12	+14	−6			
$a_1^3a_3a_5$		*	*	*	+112	−288	−6	−8	+4			
$a_2^3a_5$		*		+27	+54	−270		−9	+4	+12		
$a_3a_4^2$		+35	−15	−5	−60	+60	+2	+2	−1	−1	+1	
$a_1a_2a_4^2$			+10	+45	+205	−405	−6	−11	+3	+3	−3	
$a_1a_3^2a_4$				−10	−20	+20	−1	−1	*	*	*	
$a_1^5a_6$				*	*	*	−2	*	*	*	*	
$a_1a_3^2a_4$				*	*	*			+3	+8	−8	
$a_2^2a_3a_4$				−45	−90	+450	−3	+14	−6	−36	+6	+1

TABLE 11A (*Continued*).

$a_2a_3^3$	+20	+40	−200	+2	−6	+2	+18		−1	+1	
$a_1^3a_4^2$		−70	+270	+4	+9	−2	−2	+2		*	
$a_1^2a_2^2a_5$			+216	−9	+6	−3	−36	*	*	*	
$\imath_1^2a_2a_3a_4$			−360	+3	−16	+6	+36	+24	−2	*	
$a_1^2a_3^3$			+160		+8	−8	−34	+16	+1	−1	
$a_1a_2^3a_4$				+9	+3	−2	+48	−18	−3	*	
$a_1a_2^2a_3^2$				−6	−2	+6	−18	−24	+5	−9	
$a_2^4a_3$						−1	−3	+9	−1	+4	+1
$a_1^4a_2a_5$				+6			+36	*	*	*	*
$a_1^4a_3a_4$				−2		−4	−14	−16	+1	*	*
$a_1^3a_2^2a_4$				−6		+3	−72	+12	+8	*	*
$a_1^3a_2a_3^2$				+4		+8	+78	−48	−7	+15	
$a_1^2a_2^3a_3$						−10	−30	+72	−5	+11	−4
$a_1a_2^5$						+3	+9	−27	+3	−12	−3
$a_1^6a_5$							−12	*	*	*	*
$a_1^5a_2a_4$							+30		−7	*	*
$a_1^5a_3^2$							−20	+32	+2	−6	
$a_1^4a_2^2a_3$								−48	+10	−39	+6
$a_1^3a_2^4$								+18	−5	+29	+14
$a_1^7a_4$									+2	*	*
$a_1^6a_2a_3$									−4	+32	−4
$a_1^5a_2^3$									+2	−23	−26
$a_1^8a_3$										−8	+1
$a_1^7a_2^2$										+6	+24
$a_1^9a_2$											−11
a_1^{11}											+2

TABLE 11B.

a_{11}	+11											
a_1a_{10}	−11											
a_2a_9	+7	+12										
$a_1^2a_9$	+2	−6										
a_3a_8	−5	−18	+42									
$a_1a_2a_8$	−2	+6	−42									
$a_1^3a_8$	*	*	+14									
a_4a_7	+3	+14	−42	+14								
$a_1a_3a_7$	+2	+4		−14								
$a_2^2a_7$		−5	+35	+7	+56							
$a_1^2a_2a_7$	*	*	−14		−56							
$a_1^4a_7$	*	*	*	*	+14							
a_5a_6	−1	−5	+15	−10	−15	+5						
$a_1a_4a_6$	−2	−9	+27	−4	+15	−5						
$a_2a_3a_6$		+5	−39	−2	−87	+1	+30					
$a_1^2a_3a_6$	*	*	+6	+10	+36	+2	−15					
$a_1a_2^2a_6$	*	*	+4	−5	+31	−1	−30					
$a_1^3a_2a_6$	*	*	*	*	−14		+25					
$a_1a_5^2$	+1	+5	−15	+10	+15	−5						
$a_2a_4a_5$		−1	+7	−4	−5	+15		+20				
$a_1^2a_4a_5$		*	−2	−6	−20			−10				
$a_3^2a_5$		−2	+18	+5	+75	−25	−45	−45	+45			
$a_1a_2a_3a_5$		*	−4	−4	−58	+34	+60	+70	−90			
$a_1^3a_3a_5$		*	*	*	+14	−12	−15	−20	+30			
$a_2^3a_5$		*		+3	+9	−15		−30	+40	+80		
$a_3a_4^2$		+1	−9	−2	−36	+12	+24	+24	−36	−24	+12	
$a_1a_2a_4^2$			+2	+6	+41	−27	−24	−44	+36	+24	−12	
$a_1a_3^2a_4$				−1	−3	+1	−3	−3		*	*	
$a_1^5a_6$				*	*	*	−5	*		*	*	
$a_1a_3^2a_4$				*	*	*			+27	+48	−24	
$a_2^2a_3a_4$				−8	−8	+15	−6	+28	−36	−144	+12	+72

Table 11B (*Continued*).

$a_2a_3^3$	+1	+3	—5	+3	—9	+9	+54		—54	+54	
$a_1^3a_4^2$		—7	+9	+8	+18	—12	—8	+4		*	
$a_1^2a_2^2a_5$			+6	—15	+10	—15	—120	*	*	*	
$a_1^2a_2a_3a_4$			—6	+3	—16	+18	+72	+24	—72	*	
$a_1^2a_3^3$			+2		+6	—18	—51	+12	+27	—27	
$a_1a_2^3a_4$				+6	+2	—4	+64	—12	—72		
$a_1a_2^2a_3^2$				—3	—1	+9	—18	—12	+90	—162	
$a_2^4a_3$						—1	—2	+3	—12	+48	+48
$a_1^4a_2a_5$				+5			+60	*	*	*	*
$a_1^4a_3a_4$				—1		—6	—14	—8	+18	*	*
$a_1^3a_2^2a_4$				—2		+3	—48	+4	+96	*	*
$a_1^3a_2a_3^2$				+1		+6	+39	—12	—63	+135	
$a_1^2a_2^3a_3$						—5	—10	+12	—30	+66	—96
$a_1a_2^5$						+1	+2	—3	+12	—48	—48
$a_1^6a_5$							—10	*	*	*	*
$a_1^5a_2a_4$							+10		—42	*	*
$a_1^5a_3^2$							—5	+4	+9	—27	
$a_1^4a_2^2a_3$								—4	+30	—117	+72
$a_1^3a_2^4$								+1	—10	+58	+112
$a_1^7a_4$									+6	*	*
$a_1^6a_2a_3$									—6	+48	—24
$a_1^5a_2^3$									+2	—23	—104
$a_1^8a_3$										—6	+3
$a_1^7a_2^2$										+3	+48
$a_1^9a_2$											—11
a_1^{11}											+1

TABLE 12A.

a_{12}	+1															
a_1a_{11}	—12															
a_2a_{10}	+66	+3														
a_3a_9	—220	—15	+15													
a_4a_8	+495	+40	—40	+1												
a_5a_7	—792	—70	+70	—4	25											
a_6^2	+462	+42	—42	+3	—24	+1										
$a_1^2a_{10}$		—3	*	*	*	*										
$a_1a_2a_9$		+15	—45	*	*	*										
$a_1a_3a_8$		—25	+25	—4	*	*										
$a_2^2a_8$		—15	+150	+3	*	*	+1									
$a_1a_4a_7$		+30	—30	+12	—125	*										
$a_2a_3a_7$		+40	—400	—8	+50	*	—4	+4								
$a_1a_5a_6$		—14	+18	—8	+113	—12										
$a_2a_4a_6$		—70	+700	—22	+680	—70	+8	—8	+1							
$a_3^2a_6$				+24	—570	+80			—1	+5						
$a_2a_5^2$		+42	—420	+24	—675	+100	—5	+5	—1	+2	+1					
$a_3a_4a_5$				—36	+925	—200			+2	—19	—2	+18				
a_4^3				+15	—400	+100			—1	+12	+1	—17	+1			
$a_1^3a_9$			+30		*	*	*	*	*	*	*	*	*			
$a_1^2a_2a_8$			—135		*	*	—2	*	*	*	*	*	*			
$a_1^2a_3a_7$			+360		+200	*	+4	—4	*	*	*	*	*			
$a_1a_2^2a_7$					—150	*	+4	—12	*	*	*	*	*			
$a_1^2a_4a_6$			—630		—525	+100	—8	+8	—1		*	*	*			
$a_1a_2a_3a_6$					+350	—200	—4	+4	+2	—30	*	*	*			
$a_2^3a_6$						+100	—4	+32	—1	+20	*	*	*	+1		
$a_1^2a_5^2$			+378		+336	—64	+5	—5	+1	—2	—1		*			
$a_1a_2a_4a_5$					—105	+20	+2	—2	—2	+37	—4	—54	*			
$a_1a_3^2a_5$					—280	+320			—2	+46	+8	—72	*			
$a_2^2a_3a_5$						—200	+8	—64	+2	—49	—2	+54	*	—3	+3	
$a_1a_3a_4^2$					+175	—200			+2	—41	—2	+114	—12			
$a_1^4a_8$						*	+1	*	*	*	*	*	*	*	*	
$a_1^3a_2a_7$						*	—4	+20	*	*	*	*	*	*	*	
$a_1^3a_3a_6$						*	+4	—4		+20	*	*	*		*	
$a_1^2a_2^2a_6$						*	+4	—60		—15	*	*	*	—3	*	
$a_1^3a_4a_5$						*	—2	+2		—18	+6	+36	*			
$a_1^5a_7$						*	*	—8	*	*	*	*	*	*	*	
$a_1^2a_2a_3a_5$						*	—8	+120	*	*	*	+216	*	*	*	
$a_1^4a_2a_6$						*	*	+28	*	*	*	*	*	*	*	
$a_2^2a_4^2$						+125	—5	+40	+1	—32	+14	+162	—18	+2	—2	+1
$a_1^4a_2a_6$							*	*	*	*	*	*	*	+3	*	*
$a_1^2a_2a_3a_5$							*	*	*	—24	—12	*	*	+6	—6	*

TABLE 12A (*Continued*).

$a_1^6 a_6$	*	*	*	*	*	*	*	−1	*	*					
$a_1 a_2^3 a_5$			*	+27	+6	−162		+3	−9	*					
$a_1^4 a_3 a_5$	*	−56	*	*	*	−144		−3	+3	*					
$a_1^3 a_2^2 a_5$	*		*	*		+108		−6	+24	*					
$a_1^5 a_2 a_5$	*	*	*	*	*	*	*	+3	−21	*					
$a_1^7 a_5$	*	*	*	*	*	*	*	*	+6	*					
$a_2 a_3^2 a_4$			−3	+91	−22	−342	+54			−2	+1				
$a_1^2 a_2 a_4^2$	+5	−75	*	+45	−15	−360	+54	−4	+4	−2					
$a_1^2 a_3^2 a_4$			*	−10	+10	+66	−6			+2	−1				
$\imath_1 a_2^2 a_3 a_4$			*	−45	+30	+486	−180	−1	+1	+4	−6				
$a_2^4 a_4$			*		−15	−81	+81	−3	+18	−2	+4		+1		
$a_1^4 a_4^2$		+35	*	*	*	+135	−27	+2	−2	+1					
$\imath_1^3 a_2 a_3 a_4$			*	*	*	−180	+108	+2	−2	−4	+10				
$a_1^2 a_2^3 a_4$			*	*	*		−54	+6	−51	+2	−7		−4		
$a_1^5 a_3 a_4$			*	*	*	*	*	−1	+1	*	−4				
$a_1^4 a_2^2 a_4$			*	*	*	*	*	−3	+48	*	+3		+6		
$a_1^6 a_2 a_4$			*	*	*	*	*	*	−15	*	*		−4		
$a_1^8 a_4$			*	*	*	*	*	*	*	*	*		+1		
a_3^4			+1	−32	+9	+135	−27			+1	−1	+1			
$a_1 a_2 a_3^3$				+20	−20	−252	+108			−4	+8	−12			
$a_2^3 a_3^2$					+10	+54	−54	+2	−12	+2	−5	+8	−1	+1	
$a_1^3 a_3^3$						+80	−64				−4	+8			
$a_1^2 a_2^2 a_3^2$							+36	−4	+34	+4	−9	+30	+3	−3	
$a_1 a_2^4 a_3$										−4	+14	−48	+2	−6	
$a_1^4 a_2 a_3^2$								+2	−32	*	+8	−48	−3	+3	
$a_1^3 a_2^3 a_3$										*	−10	+68	−6	+22	
$a_1^6 a_3^2$									+10	*	*	+16	+1	−1	
$a_1^5 a_2^2 a_3$										*	*	−24	+6	−30	
$a_1^7 a_2 a_3$										*	*	*	−2	+18	
$a_1^9 a_3$										*	*	*	*	−4	
a_2^6										+1	−4	+16	−1	+4	+1
$a_1^2 a_2^5$											+3	−24	+3	−15	−6
$a_1^4 a_2^4$												+9	−3	+21	+15
$a_1^6 a_2^3$													+1	−13	−20
$a_1^8 a_2^2$														+3	+15
$a_1^{10} a_2$															−6
a_1^{12}															+1

TABLE 12B.

a_{12}	+2															
a_1a_{11}	—2															
a_2a_{10}	+2	+18														
a_3a_9	—2	—27	+378													
a_4a_8	+2	+82	—428	+28												
a_5a_7	—2	—35	+490	—70	+525											
a_6^2	+1	+18	—252	+45	—432	+9										
$a_1^2a_{10}$		—9	*	*	*	*										
$a_1a_2a_9$		+9	—378	*	*	*										
$a_1a_3a_8$		—5	+70	—28	*	*										
$a_2^2a_8$		—2	+280	+28	*	*	+56									
$a_1a_4a_7$		+3	—42	+42	—525	*										
$a_2a_3a_7$		+2	—280	—14	+175	*	—21	+84								
$a_1a_5a_6$		—1	+18	—20	+339	—18										
$a_2a_4a_6$		—2	+280	—22	+816	—42	+96	—96	+240							
$a_3^2a_6$				+18	—513	+36			—180	+225						
$a_2a_5^2$		+1	—140	+20	—675	+50	—50	+50	—200	+100	+200					
$a_3a_4a_5$				—18	+555	—60			+240	—570	—240	+720				
a_4^3				+6	—192	+24			—96	+288	+96	—544	+32			
$a_1^3a_9$			+126		*	*	*	*	*	*	*	*	*			
$a_1^2a_2a_6$			—126		*	*	—56	*	*	*	*	*	*			
$a_1^2a_3a_7$			+126		+210	*	+168	—42	*	*	*	*	*			
$a_1a_2^2a_7$					—105	*	+28	—84	*	*	*	*	*			
$a_1^2a_4a_6$			—126		—315	+30	—48	+48	—120		*	*	*			
$a_1a_2a_3a_6$					+105	—30	—12	+12	+120	—450	*	*	*			
$a_2^3a_6$						+10	—8	+64	—40	+200	*	*	*	+40		
$a_1^2a_5^2$			+63		+168	—16	+25	—25	+100	—50	—100		*			
$a_1a_2a_4a_5$					—21	+2	+4	—4	—80	+370	—160	—720	*			
$a_1a_3^2a_5$					—42	+24			—60	+345	+240	—720	*			
$a_2^2a_3a_5$						—10	+8	—64	+40	—245	—40	+360	*	—60	+120	
$a_1a_3a_4^2$					+21	—12			+48	—246	—48	+912	—96			
$a_1^4a_8$						*	+14	*	*	*	*	*	*	*	*	
$a_1^3a_2a_7$						*	—14	+70	*	*	*	*	*	*	*	
$a_1^3a_3a_6$						*	+6	—6		+530	*	*	*		*	
$a_1^2a_2^2a_6$						*	+4	—60		—75	*	*	*	—60	*	
$a_1^3a_4a_5$						*	—2	+2		—90	+120	+240	*			
$a_1^5a_7$						*	*	—14	*	*	*	*	*	*	*	
$a_1^2a_2a_3a_5$						*	—4	+60	*	*	*	+720	*	*	*	
$a_1^4a_2a_6$						*	*	+14	*	*	*	*	*	*	*	
$a_2^2a_4^2$						+5	—4	+32	+16	—128	+224	+864	—96	+32	—64	+144
$a_1^4a_2a_6$							*	*	*	*	*	*	*	+30	*	*
$a_1^2a_2a_3a_5$							*	*	*	—60	—120	*	*	+60	—120	*

$a_1^6a_6$	*	*	*	*	*	*	*	—5	*	*					
$a_1a_2^3a_5$			*	+45	+40	—360		+20	—120	*					
$a_1^4a_3a_5$	*	—14	*	*	*	—240		—15	+30	*					
$a_1^3a_2^2a_5$	*		*	*		+120		—20	+160	*					
$a_1^5a_2a_5$	*	*	*	*	*	*	*	+5	—70	*					
$a_1^7a_5$	*	*	*	*	*	*	*	*	+10	*					
$a_2a_3^2a_4$			—36	+273	—132	—1368	+216			—216	+108				
$a_1^2a_2a_4^2$	+2	—30	*	+90	—120	—960	+144	—32	+64	—144					
$a_1^2a_3^2a_4$			*	—15	+60	+132	—12			+108	—54				
$a_1a_2^2a_3a_4$			*	—45	+120	+648	—240	—4	+8	+144	—216				
$a_2^4a_4$			*		—40	—72	+72	—8	+96	—48	+96		+96		
$a_1^4a_4^2$		+7	*	*	*	+180	—36	+8	—16	+36					
$a_1^3a_2a_3a_4$			*	*	*	—120	+72	+4	—8	—72	+180				
$a_1^2a_2^3a_4$			*	*	*		—24	+8	—136	+24	—84		—192		
$a_1^5a_3a_4$			*	*	*	*	*	—1	+2	*	—36				
$a_1^4a_2^2a_4$			*	*	*	*	*	—2	+64	*	+18		+144		
$a_1^6a_2a_4$			*	*	*	*	*	*	—10	*	*		—48		
$a_1^8a_4$			*	*	*	*	*	*	*	*	*		+6		
a_3^4			+9	—72	+81	+405	—81			+81	—81	+81			
$a_1a_2a_3^3$				+15	—60	—252	+108			—108	+216	—324			
$a_2^3a_3^2$					+20	+36	—36	+4	—48	+36	—90	+144	—72	+72	
$a_1^3a_3^3$						+40	—32				—54	+108			
$a_1^2a_2^2a_3^2$							+12	—4	+68	+36	—81	+270	+108	—108	
$a_1a_2^4a_3$										—24	+84	—288	+48	—144	
$a_1^4a_2a_3^2$								+1	—32	*	+36	—216	—54	+54	
$a_1^3a_2^3a_3$										*	—30	+204	—72	+264	
$a_1^6a_3^2$									+5	*	*	+36	+9	—9	
$a_1^5a_2^2a_3$										*	*	—36	+36	—180	
$a_1^7a_2a_3$										*	*	*	—6	+54	
$a_1^9a_3$										*	*	*	*	—6	
a_2^6										+4	—16	+64	—16	+64	+64
$a_1^2a_2^5$											+6	—48	+24	—120	—192
$a_1^4a_2^4$												+9	—12	+84	+240
$a_1^6a_2^3$													+2	—26	—160
$a_1^8a_2^2$														+3	+60
$a_1^{10}a_2$															—12
a_1^{12}															+1

TABLE 12B (*Continued*).